

# Bi-invariant metric on volume-preserving diffeomorphisms group of a three-dimensional manifold

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## Abstract

We show the existence of a weak bi-invariant symmetric nondegenerate 2-form on the volume-preserving diffeomorphism group of a three-dimensional manifold and study its properties. Despite the fact that the space  $\mathcal{D}_\mu(M^3)$  is infinite-dimensional, we succeed in defining the signature of the bi-invariant quadric form. It is equal to the  $\eta$ -invariant of the manifold  $M^3$ .

## 1 Invariant form on $d\Gamma(\Lambda^{2k-1}M)$

Let  $(M, g)$  be a smooth (of class  $C^\infty$ ) compact Riemannian manifold of dimension  $n = 4k - 1$  without boundary. Consider the elliptic self-adjoint operator  $A$  acting on smooth exterior differential forms on  $M$  of even degree  $2p$ , given by

$$A\varphi = (-1)^{k+p+1}(*d - d*)\varphi, \quad \deg \varphi = 2p.$$

Here  $d$  is exterior differential and  $*$  is the Hodge duality operator defined by metric. Let  $\delta = (-1)^{np+n+1} * d*$  is the codifferential acting on exterior differential  $p$ -forms on  $M$ .

The space  $\Gamma(\Lambda^{2k}M)$  of smooth forms on  $M$  of degree  $2k$  is invariant under the action of  $A$ . Let us write the Hodge decomposition of the space  $\Gamma(\Lambda^{2k}M)$  into a direct sum:

$$\Gamma(\Lambda^{2k}M) = \delta\Gamma(\Lambda^{2k+1}M) \oplus H^{2k} \oplus d\Gamma(\Lambda^{2k-1}M),$$

where  $\delta\Gamma(\Lambda^{2k+1}M)$  is the space of co-exact forms and  $H^{2k}$  is the space of harmonic forms, and let  $d\Gamma(\Lambda^{2k-1}M)$  be the space of exact  $2k$ -forms. The kernel of the operator  $A = (-1)^{2k+1}(*d - d*) = -*d + d*$  on the space  $\Gamma(\Lambda^{2k}M)$  consists of harmonic forms and the image obviously coincides with the space of exact and co-exact forms. Moreover, the first and second components of the operator  $A = -*d + d*$  separately act on  $\delta\Gamma(\Lambda^{2k+1}M)$  и  $d\Gamma(\Lambda^{2k-1}M)$ :

$$\begin{aligned} \text{Im}(-*d) &= \delta\Gamma(\Lambda^{2k+1}M), & \text{Ker}(-*d) &= H^{2k} \oplus d\Gamma(\Lambda^{2k-1}M), \\ \text{Im}(d*) &= d\Gamma(\Lambda^{2k-1}M), & \text{Ker}(d*) &= H^{2k} \oplus \delta\Gamma(\Lambda^{2k+1}M). \end{aligned}$$

Therefore, the restriction of the operator  $A = -*d + d*$  to the direct sum  $\delta\Gamma(\Lambda^{2k+1}M) \oplus d\Gamma(\Lambda^{2k-1}M)$  is an isomorphism preserving this decomposition. Hence the operator  $A$  has an inverse  $A^{-1}$  on the space  $\delta\Gamma(\Lambda^{2k+1}M)$ , as well as on the space  $d\Gamma(\Lambda^{2k-1}M)$ . In what follows, we use the operator

$$A^{-1} : d\Gamma(\Lambda^{2k-1}M) \rightarrow d\Gamma(\Lambda^{2k-1}M)$$

inverse to the operator  $A = d*$  on the space  $d\Gamma(\Lambda^{2k-1}M)$ .

On the space  $\Gamma(\Lambda^{2k}M)$ , there exists the natural inner product

$$(\alpha, \beta) = \int_M \alpha \wedge *\beta, \quad \alpha, \beta \in \Gamma(\Lambda^{2k}M). \quad (1.1)$$

Then the Hodge decomposition  $\Gamma(\Lambda^{2k}M) = \delta\Gamma(\Lambda^{2k+1}M) \oplus H^{2k} \oplus d\Gamma(\Lambda^{2k-1}M)$  is orthogonal. Let  $p : \Gamma(\Lambda^{2k}M) \rightarrow d\Gamma(\Lambda^{2k-1}M)$  be the orthogonal projection.

Let  $L_X = d \cdot i_X + i_X \cdot d$  be the Lie derivative, where  $i_X$  is the inner product,  $i_X \alpha = \alpha(X, \cdot)$ .

**Lemma 1.1.** *On the space  $d\Gamma(\Lambda^{2k-1}M)$  the following equality is fulfilled*

$$A^{-1} \cdot L_X = p * i_X$$

for any vector field  $X$ .

*Proof.* Let  $d\alpha \in d\Gamma(\Lambda^{2k-1}M)$ . Then  $*i_X d\alpha = d\beta + \gamma$  where  $d\beta \in d\Gamma(\Lambda^{2k-1}M)$  and  $\gamma \in \text{Ker}\delta = \text{Ker}p = \text{Ker}(d*)$ . We will notice that on the space  $d\Gamma(\Lambda^{2k-1}M)$  we have  $L_X = d \cdot i_X$  and  $A = d*$ . Therefore,

$$\begin{aligned} p * i_X(d\alpha) &= d\beta, \\ A^{-1} \cdot L_X(d\alpha) &= (d*)^{-1} d i_X(d\alpha) = (d*)^{-1} d * i_X(d\alpha) = \\ &= (d*)^{-1} (d*)(d\beta + \gamma) = (d*)^{-1} (d*)(d\beta) = d\beta. \end{aligned}$$

□

**Lemma 1.2.** *The operator  $p * i_X$  is skew-symmetric on the space  $d\Gamma(\Lambda^{2k-1}M)$ :*

$$(p * i_X d\alpha, d\beta) + (d\alpha, p * i_X d\beta) = 0$$

for all  $d\alpha, d\beta \in d\Gamma(\Lambda^{2k-1}M)$ .

*Proof.* As the operator  $p$  is orthogonal projection on  $d\Gamma(\Lambda^{2k-1}M)$ , then

$$\begin{aligned} (p * i_X d\alpha, d\beta) + (d\alpha, p * i_X d\beta) &= (*i_X d\alpha, d\beta) + (d\alpha, *i_X d\beta) = \\ &= \int_M i_X d\alpha \wedge d\beta + \int_M d\alpha \wedge * * i_X d\beta = \int_M (i_X d\alpha \wedge d\beta + d\alpha \wedge i_X d\beta) = 0, \end{aligned}$$

as  $i_X d\alpha \wedge d\beta + d\alpha \wedge i_X d\beta = i_X(d\alpha \wedge d\beta) = 0$ . □

Consider the following bilinear form  $Q$  on  $d\Gamma(\Lambda^{2k-1}M)$  introduced in [5]: for  $d\alpha, d\beta \in d\Gamma(\Lambda^{2k-1}M)$ ,

$$Q(d\alpha, d\beta) = (d\alpha, A^{-1}d\beta) = \int_M d\alpha \wedge \beta. \quad (1.2)$$

Since the operator  $A$  is self-adjoint, the form  $Q$  is symmetric. The signature of the corresponding quadratic form  $Q(d\alpha, d\alpha)$  is equal to the  $\eta$ -invariant of the manifold  $M$  [5]. If  $M$  is the boundary of a  $4k$ -dimensional manifold  $N$ , then (see [5])

$$\eta = \int_N L_k - \text{sign}N,$$

where  $L_k = L_k(p_1, \dots, p_k)$  is the  $L$ -Hirzebruch polynomial and  $\text{sign}N$  is the signature of the natural quadratic form on the  $2k$ -cohomology space  $H^{2k}(N, \mathbb{R})$ .

Let  $\mathcal{D}_0$  be the connected components of the identity of the smooth diffeomorphism group of the manifold  $M$ . The group  $\mathcal{D}_0$  acts on the space  $d\Gamma(\Lambda^{2k-1}M)$  to the right:

$$d\Gamma(\Lambda^{2k-1}M) \times \mathcal{D}_0 \rightarrow d\Gamma(\Lambda^{2k-1}M), \quad (d\alpha, \eta) \rightarrow \eta^*(d\alpha),$$

where  $\eta^*$  is the codifferential of a diffeomorphism  $\eta \in \mathcal{D}_0$ . Expression (1.2) for the quadratic form  $Q$  implies the following:

**Theorem 1.3.** *The bilinear form  $Q$  on  $d\Gamma(\Lambda^{2k-1}M)$  is invariant under the action of the group  $\mathcal{D}_0$  on  $d\Gamma(\Lambda^{2k-1}M)$ : for any vector field  $X$  on  $M$ ,*

$$Q(L_X d\alpha, d\beta) + Q(d\alpha, L_X d\beta) = 0. \quad (1.3)$$

*Proof.*

$$\begin{aligned} Q(L_X d\alpha, d\beta) + Q(d\alpha, L_X d\beta) &= (L_X d\alpha, A^{-1}d\beta) + (d\alpha, A^{-1}L_X d\beta) = \\ &= (AA^{-1}L_X d\alpha, A^{-1}d\beta) + (d\alpha, *i_X d\beta) = (A^{-1}L_X d\alpha, AA^{-1}d\beta) + (d\alpha, *i_X d\beta) = \\ &= (*i_X d\alpha, d\beta) + (d\alpha, *i_X d\beta). \end{aligned}$$

□

## 2 Bi-invariant metric on the group $\mathcal{D}_\mu(M)$

Let  $\mathcal{D}_\mu$  be the group of diffeomorphisms of manifold  $M$  leaving the Riemannian volume element  $\mu$  fixed,

$$\mathcal{D}_\mu = \{\eta \in \mathcal{D}; \eta^* \mu = \mu\}$$

where  $\eta^*$  is the codifferential of the diffeomorphism  $\eta$ . If  $X$  is a vector field on  $M$ , then the divergence  $\text{div} X$  of the field  $X$  is defined by  $L_X \mu = (\text{div} X) \mu$ , where  $L_X$  is the Lie derivative. If  $\eta_t$  is a one-parametric subgroup of the group  $\mathcal{D}_\mu$ , then  $\eta_t^* \mu = \mu$ . Differentiating this relation in  $t$ , we obtain  $L_X \mu = 0$  or  $\text{div} X = 0$ , where  $X$  is the vector field of velocities of the flow  $\eta_t$  on  $M$ . Therefore, the Lie algebra of the group  $\mathcal{D}_\mu$  consists of divergence-free vector fields, i.e., those fields  $X$  for which  $\text{div}_\mu X = 0$ . Ebin and Marsden showed in [7] that the group  $\mathcal{D}_\mu$  is a closed ILH-subgroup of the ILH-Lie group  $\mathcal{D}$  with the Lie algebra  $T_e \mathcal{D}_\mu$  consisting of divergence-free vector fields on  $M$ . The formula

$$(X, Y) = \int_M g(X, Y) d\mu, \quad X, Y \in T_e \mathcal{D}_\mu \quad (2.1)$$

is defined on  $\mathcal{D}_\mu$  a smooth right-invariant weak Riemannian structure. In [1] and [7] it was shown that geodesics on the group  $\mathcal{D}_\mu$  are flows of the ideal incompressible fluid. (At integration the form  $\mu$  we will note as  $d\mu$ ).

On a Riemannian manifold, there exists a natural isomorphism between the space  $\Gamma(TM)$  of smooth vector fields on  $M$  and the space  $\Gamma(\Lambda^1 M)$  of smooth 1-forms. To each vector field  $Y$ , it puts in correspondence the 1-form  $\omega_Y$  such that  $\omega_Y(X) = g(Y, X)$ , where  $g$  is the Riemannian metric on  $M$ . If  $\dim M = 3$ , then there exists one more isomorphism between  $\Gamma(TM)$  and the space of 2-forms on  $M$ . To a vector field  $X$  on  $M$ , it puts in correspondence the 2-form  $i_X \mu$ , where  $\mu$  is the Riemannian volume element on  $M$  and  $i_X$  is the inner product. Then the sequence of operators grad, rot, div,

$$0 \rightarrow C^\infty(M, \mathbb{R}) \rightarrow \Gamma(TM) \rightarrow \Gamma(TM) \rightarrow C^\infty(M, \mathbb{R}) \rightarrow 0.$$

corresponds to the sequence of exterior differentials

$$0 \rightarrow C^\infty(M, \mathbb{R}) \rightarrow \Gamma(\Lambda^1 M) \rightarrow \Gamma(\Lambda^2 M) \rightarrow \Gamma(\Lambda^3 M) \rightarrow 0$$

The operator rot is defined by  $d\omega_X = i_{\text{rot} X} \mu$ .

We see from the relation  $di_X \mu = L_X \mu = (\text{div} X) \mu$  that the 2-form  $i_X \mu$  is closed iff  $\text{div} X = 0$ . Clearly, the 2-form  $i_X \mu$  is exact iff  $X = \text{rot} V$  and  $i_X \mu = d\omega_V = i_{\text{rot} V} \mu$ . The space  $d\Gamma(\Lambda^1 M)$  is isomorphic to the subspace  $\text{Im}(\text{rot}) \subset \Gamma(TM)$ . Note that

$$(\omega_Y, \omega_X) = (i_Y \mu, i_X \mu) = (Y, X) = \int_M g(Y, X) d\mu$$

and  $*\omega_Y = i_Y \mu$ ,  $*i_X \mu = \omega_X$ . This immediately implies that the operator  $\text{rot} : \text{Im}(\text{rot}) \rightarrow \text{Im}(\text{rot})$  corresponds to the operator  $A = d* : d\Gamma(\Lambda^1 M) \rightarrow d\Gamma(\Lambda^1 M)$ . Indeed, each exact 2-form  $d\alpha \in d\Gamma(\Lambda^1 M)$  is represented in the form  $d\alpha = i_X \mu$ , where  $X \in \text{Im}(\text{rot})$ ; therefore,

$$A(i_X \mu) = (d*)(i_X \mu) = d(\omega_X) = i_{\text{rot} X} \mu.$$

The operator  $A^{-1}$  corresponds to the operator  $\text{rot}^{-1}$ ,  $A^{-1}(i_X \mu) = i_{\text{rot}^{-1} X} \mu$ .

We have represented every 2-form  $d\alpha \in d\Gamma(\Lambda^1 M)$  in the form  $d\alpha = i_X \mu$ ,  $X \in \text{Im}(\text{rot})$ . Let us find the corresponding expression of the symmetric 2-form  $Q$  on the space  $d\Gamma(\Lambda^1 M)$ :

$$Q(d\alpha, d\beta) = Q(i_X \mu, i_Y \mu) = (i_X \mu, A^{-1} i_Y \mu) = (i_X \mu, i_{\text{rot}^{-1} Y} \mu) =$$

$$= (X, \text{rot}^{-1}Y) = \int_M g(X, \text{rot}^{-1}Y) d\mu.$$

Therefore, to the bilinear form  $Q$  on  $d\Gamma(\Lambda^1 M)$ , we put in correspondence the following bilinear symmetric form on  $\text{Im}(\text{rot})$ :

$$\langle X, Y \rangle_e = Q(i_X \mu, i_Y \mu) = \int_M g(X, \text{rot}^{-1}Y) d\mu, \quad (2.2)$$

where  $X, Y \in \text{Im}(\text{rot}) \subset \Gamma(TM)$ . This form is nondegenerate; indeed, for any  $X \in \text{Im}(\text{rot})$ , we have  $\langle X, \text{rot}X \rangle = (X, X) > 0$  if  $X \neq 0$ . The symmetry of the form (2.2) follows from the self-adjointness of the operator  $\text{rot} : \Gamma(TM) \rightarrow \Gamma(TM)$ .

The space  $\text{Im}(\text{rot})$  is a Lie subalgebra of the Lie algebra  $T_e \mathcal{D}_\mu$ , and, moreover, it is its ideal. This follows from the fact that the Lie bracket  $[X, Y]$  of divergence-free vector fields defines the exact 2-form  $i_{[X, Y]} \mu = d(i_X i_Y \mu)$ . The algebra  $T_e \mathcal{D}_\mu$  of divergence-free vector fields on  $M$  consists of all vector fields  $X$  on  $M$  for which the  $(n-1)$ -form  $i_X \mu$  is closed:  $L_X \mu = d(i_X \mu) = 0$ . The algebra  $\text{Im}(\text{rot})$  consists of all vector fields  $X$  for which the  $(n-1)$ -form  $i_X \mu$  is exact. Such vector fields  $X$  are said to be exact divergence-free.

In [16], it was shown that there exists an ILH-Lie group  $\mathcal{D}_{\mu\partial} \subset \mathcal{D}_\mu$  whose algebra Lie is the algebra of exact divergence-free vector fields on  $M$ . The group  $\mathcal{D}_{\mu\partial}$  is called the group of exact diffeomorphisms preserving the volume element  $\mu$ . Taking this fact into account, in what follows, we denote the space  $\text{Im}(\text{rot})$  by  $T_e \mathcal{D}_{\mu\partial}$  and consider it as the Lie algebra of the group  $\mathcal{D}_{\mu\partial}$ . The algebra  $T_e \mathcal{D}_{\mu\partial}$  differs from the algebra  $T_e \mathcal{D}_\mu$  by the cohomology space  $H^2(M, \mathbb{R})$ :

$$T_e \mathcal{D}_\mu = T_e \mathcal{D}_{\mu\partial} \oplus H^2(M, \mathbb{R}).$$

If  $H^2(M, \mathbb{R}) = 0$ , then the groups  $\mathcal{D}_\mu$  and  $\mathcal{D}_{\mu\partial}$  and their Lie algebras coincide. Therefore, the form

$$\langle X, Y \rangle_e = \int_M g(X, \text{rot}^{-1}Y) d\mu(x)$$

is a bilinear symmetric nondegenerate form on the Lie algebra  $T_e \mathcal{D}_{\mu\partial}$  of the group  $\mathcal{D}_{\mu\partial}$  of the exact diffeomorphisms preserving the volume element  $\mu$ .

Using right translations, the symmetric 2-form (2.2) defines the following right-invariant symmetric 2-form on the whole group  $\mathcal{D}_{\mu\partial}$ : for  $X_\eta, Y_\eta \in T_\eta \mathcal{D}_{\mu\partial}$ ,

$$\langle X_\eta, Y_\eta \rangle_\eta = \langle dR_\eta^{-1} X, dR_\eta^{-1} Y \rangle_e, \quad \eta \in \mathcal{D}_{\mu\partial}.$$

The following theorem states that the obtained form on  $\mathcal{D}_{\mu\partial}$  is smooth and bi-invariant.

**Theorem 2.1** ([17]). *The bilinear form (2.2) on the Lie algebra  $T_e \mathcal{D}_{\mu\partial}$  of the group  $\mathcal{D}_{\mu\partial}$  defines the ILH-smooth bi-invariant form on the ILH-Lie group  $\mathcal{D}_{\mu\partial}$ . In particular, the following relation holds for any  $X, Y, Z \in T_e \mathcal{D}_{\mu\partial}$ :*

$$\langle [X, Y], Z \rangle_e = -\langle Y, [X, Y] \rangle_e. \quad (2.3)$$

*The signature of the quadratic form corresponding to (2.2) is equal to the  $\eta$ -invariant of the manifold  $M$ .*

*Proof.* To prove the ILH-smoothness of the form on  $\mathcal{D}_{\mu\partial}$  obtained from (2.2) by right translations, we use the Omori result [15] on the smoothness of the right-invariant morphism  $T\mathcal{D}^s \rightarrow \gamma^{s, s-1}$ ,  $s \geq n + 5 + r$ , defined by the kernel of a differential operator of order  $r$  with smooth coefficients.

The bi-invariance property  $\langle Ad_\eta X, Ad_\eta Y \rangle_e = \langle X, Y \rangle_e$ ,  $\eta \in \mathcal{D}_{\mu\partial}$ , of the form  $\langle X, Y \rangle_e = Q(i_X \mu, i_Y \mu)$  follows from the  $\mathcal{D}$ -invariance of the form  $Q$  and the fact that  $\eta^*(i_X \mu) = -i_{Ad_\eta X} \mu$ ,  $\eta \in \mathcal{D}_\mu$ .

As the group  $\mathcal{D}_{\mu\partial}$  is connected, then bi-invariance property  $\langle Ad_\eta X, Ad_\eta Y \rangle_e = \langle X, Y \rangle_e$ ,  $\eta \in \mathcal{D}_{\mu\partial}$ , of the form (2.2) follows from (2.3). From  $\mathcal{D}$ -invariance of the form  $Q$  on  $d\Gamma(\Lambda^1 M)$  we have:

$$Q(L_X(i_Y \mu), i_Z \mu) + Q(i_Y \mu, L_X(i_Z \mu)) = 0.$$

From  $L_X \mu = 0$  and  $[L_X, i_Y] = i_{[X, Y]}$  we have:

$$L_X(i_Y \mu) = L_X i_Y \mu - i_Y L_X \mu = i_{[X, Y]} \mu.$$

Then, from  $\langle X, Y \rangle_e = Q(i_X \mu, i_Y \mu)$  we have:

$$\begin{aligned} 0 &= Q(L_X(i_Y \mu), i_Z \mu) + Q(i_Y \mu, L_X(i_Z \mu)) = Q(i_{[X, Y]} \mu, i_Z \mu) + Q(i_Y \mu, i_{[X, Z]} \mu) = \\ &= \langle [X, Y], Z \rangle_e + \langle Y, [X, Z] \rangle_e. \end{aligned}$$

The operator  $\text{rot}$  is not positive-definite: its eigenvalues  $\lambda_i$  can be positive as well as negative (the squares of these eigenvalues are the eigenvalues of the Laplace operator  $\Delta = \text{rot} \circ \text{rot}$ ). Therefore, the quadratic form  $\langle X, X \rangle_e = (X, \text{rot}^{-1} X)_e$  is not positive-definite. The signature of this quadratic form that is understood as the limit of the function

$$\eta(s) = \sum_i (\text{sign} \lambda_i) |\lambda_i|^{-s}$$

at zero is finite and is equal to the  $\eta$ -invariant of the manifold  $M$ . This follows from the fact that the  $\eta$ -invariant is equal to the signature of the form  $Q$  on  $d\Gamma(\Lambda^1 M)$  [5].  $\square$

**Remark 2.2.** If  $H^2(M, \mathbb{R}) = 0$ , then  $\mathcal{D}_\mu = \mathcal{D}_{\mu\partial}$ . In this case, form (2.2) is a bilinear symmetric form on the algebra of divergence-free vectors on  $M$  is bi-invariant with respect to the group  $\mathcal{D}_\mu(M)$  of diffeomorphisms of  $M$  preserving the volume element  $\mu$ .

**Remark 2.3.** If  $\partial M \neq 0$ , then on the space  $T_e \mathcal{D}_{\mu\partial}(M)$  of exact divergence-free vector fields on  $M$  tangent to the boundary, we can define the invariant form by

$$\langle X, Y \rangle_e = \frac{1}{2} ((\text{rot}^{-1} X, Y)_e + (X, \text{rot}^{-1} Y)_e), \quad (2.4)$$

where  $\text{rot}^{-1} X$  is the vector field on  $M$  tangent to  $\partial M$  such that its vorticity is equal to  $X$ . If  $H^2(M, \mathbb{R}) = 0$ , then the operator  $\text{rot}^{-1}$  is defined on  $T_e \mathcal{D}_\mu$ . By the usual calculations, we prove the  $\mathcal{D}_\mu$ -invariance of the inner product (2.4) on  $T_e \mathcal{D}_\mu$ .

**Remark 2.4.** The obtained expression (2.2) for the bi-invariant form is explained by the fact that the group exponential mapping of the diffeomorphism group is not surjective. Indeed, the new pseudo-Riemannian metric  $\langle X, Y \rangle_e = (X, \text{rot}^{-1} Y)_e$  is expressed through the Riemannian metric by using the operator  $\text{rot}^{-1}$ . Therefore, in the geodesic equations  $\ddot{\eta} = -\Gamma(\eta, \dot{\eta})$ , we have the compact operator  $\text{rot}^{-1}$  with due account for which the exponential mapping is also compact, and hence it is not surjective.

### 3 Euler equations on the Lie algebra $T_e\mathcal{D}_\mu$

Let  $\mathfrak{g}$  be a semisimple, finite-dimensional Lie algebra, and let  $H$  be a certain function on  $\mathfrak{g}$ . In [13], it was shown that the extension of the Euler equation on the Lie algebra of the group  $SO(n, \mathbb{R})$  of motions of an  $n$ -dimensional rigid body to the case of the general semisimple Lie algebra  $\mathfrak{g}$  is an equation of the form

$$\frac{d}{dt}X = [X, \text{grad}H(X)], \quad (3.1)$$

where  $X \in \mathfrak{g}$  and the gradient of the Hamiltonian function  $H$  is calculated with respect to the invariant Killing–Cartan inner product on  $\mathfrak{g}$ .

Assume that the second cohomology group of the manifold  $M^3$  is trivial:  $H^2(M, \mathbb{R}) = 0$ . As  $\mathfrak{g}$ , let us consider the Lie algebra  $T_e\mathcal{D}_\mu$  of divergence-free vector fields on the three-dimensional Riemannian manifold  $M$ . On  $T_e\mathcal{D}_\mu$ , we have the invariant nondegenerate form (2.2) and the function (kinetic energy)

$$T(V) = \frac{1}{2}(V, V)_e = \frac{1}{2} \int_M g(V, V) d\mu, \quad V \in T_e\mathcal{D}_\mu(M).$$

The function  $T$  can be written as follows in terms of the inner product (2.2):

$$T(V) = \frac{1}{2}(V, V)_e = \frac{1}{2}(V, \text{rot}^{-1}\text{rot}V)_e = \frac{1}{2} \langle V, \text{rot}V \rangle_e.$$

Perform the Legendre transform  $X = \text{rot}V$ ; then  $T(V) = T(\text{rot}^{-1}X) = \frac{1}{2} \langle \text{rot}^{-1}X, X \rangle_e$ . Consider the Hamiltonian function  $H(X) = \frac{1}{2} \langle \text{rot}^{-1}X, X \rangle_e$  on the Lie algebra  $T_e\mathcal{D}_\mu$ . The gradient of the function  $H$  with respect to the invariant inner product (2.2) is easily calculated:

$$\text{grad}H(X) = \text{rot}^{-1}X.$$

As in the finite-dimensional case, let us write the Euler equation on the Lie algebra  $T_e\mathcal{D}_\mu(M)$ :

$$\frac{d}{dt}X = [X, \text{rot}^{-1}X]. \quad (3.2)$$

Since  $X = \text{rot}V$ , it follows that in the variables  $V$ , this equation yields the Helmholtz equation [19]

$$\frac{\partial \text{rot}V}{\partial t} = [\text{rot}V, V], \quad (3.3)$$

in our case,  $H^2(M, \mathbb{R}) = 0$ , it is equivalent to the equation

$$\frac{\partial V}{\partial t} = \nabla_V V - \text{grad}p$$

of motion of the ideal incompressible fluid in  $M$ .

On the Lie algebra  $T_e\mathcal{D}_\mu$ , Eq. (3.2) or (3.3) has the following two quadratic first integrals:

$$m(X) = \langle X, X \rangle_e = (\text{rot}V, V) = \int_M g(\text{rot}V, V) d\mu,$$

$$H(X) = T(V) = \frac{1}{2} \int_M g(V, V) d\mu.$$

The first of them  $m(X)$  is naturally called the *kinetic moment*. The invariance of the function  $m$  follows from the invariance of the inner product (2.2) on  $T_e\mathcal{D}_\mu(M)$ . The second integral  $H(X)$  is the *kinetic energy*.

Since  $H(X) = \frac{1}{2} \langle X, \text{rot} X \rangle_e$ , the operator  $\text{rot} : T_e\mathcal{D}_\mu \rightarrow T_e\mathcal{D}_\mu$  is the inertia operator of our mechanical system  $(T_e\mathcal{D}_\mu, H)$ . The eigenvectors  $X_i$  of the operator  $\text{rot}$  are naturally called (analogously to the rigid body motion) the *axes of inertia*, and the eigenvalues  $\lambda_i$  of  $\text{rot}$  are called the *moments of inertia* with respect to the axes  $X_i$ .

The Euler equation (3.3) can be written in the form

$$\frac{\partial \text{rot} V}{\partial t} = -L_V \text{rot} V,$$

where  $L_V$  is the Lie derivative. Therefore [12], the vector field  $\text{rot} V(t)$  is transported by the flow  $\eta_t$  of the field  $V(t)$ :

$$\text{rot} V(t) = d\eta_t(\text{rot} V(0) \circ \eta_t^{-1}) = \text{Ad}_{\eta_t}(\text{rot} V(0)),$$

where  $V(0)$  is the initial velocity field. In mechanics, this property is called the property of the vorticity  $X = \text{rot} V$  to be carried along the fluid flow  $\eta_t$  [4]. Hence the curve  $X(t) = \text{rot} V(t)$  on the Lie algebra  $T_e\mathcal{D}_\mu$  that is a solution of the Euler equation (3.2) lies on the orbit  $\mathcal{O}(X_0) = \{\text{Ad}_\eta X(0); \eta \in \mathcal{D}_\mu\}$  of the coadjoint action of the group  $\mathcal{D}_\mu$ .

Since the kinetic moment  $m(X)$  is preserved under the motion, the orbit  $\mathcal{O}(X_0)$  lies on the "pseudo-sphere"

$$S = \{Y \in T_e\mathcal{D}_\mu; \langle Y, Y \rangle_e = r\},$$

where  $r = \langle X(0), X(0) \rangle_e$ . It is natural to expect that the critical points of the function  $H(X)$  on the orbit  $\mathcal{O}(X_0)$  are stationary motions. The orbit  $\mathcal{O}(X_0)$  is the image of the smooth mapping

$$\mathcal{D}_\mu \rightarrow T_e\mathcal{D}_\mu, \quad \eta \rightarrow \text{Ad}_\eta X_0.$$

Therefore, the tangent space  $T_Y\mathcal{O}(X_0)$  to the orbit at a point  $Y$  is

$$T_Y\mathcal{O}(X_0) = \{[W, Y]; W \in T_e\mathcal{D}_\mu\}.$$

A point  $Y \in \mathcal{O}(X_0)$  is critical for the function  $H(X)$  if  $\text{grad} H(X)$  is orthogonal to the space  $T_Y\mathcal{O}(X_0)$ . Since  $\text{rot}^{-1}Y = \text{grad} H(X)$ , the latter condition is equivalent to

$$\langle \text{rot}^{-1}Y, [W, Y] \rangle_e = 0 \quad \text{for any } W \in T_e\mathcal{D}_\mu(M).$$

The invariance of  $\langle \cdot, \cdot \rangle_e$  implies

$$\langle [Y, \text{rot}^{-1}Y], W \rangle_e = 0 \quad \text{для любого } W \in T_e\mathcal{D}_\mu(M).$$

From the nondegeneracy of the form  $\langle \cdot, \cdot \rangle_e$  on  $T_e\mathcal{D}_\mu(M)$ , we obtain that a point  $Y$  on the orbit  $\mathcal{O}(X_0)$  is a critical point of the function  $H$  iff

$$[Y, \text{rot}^{-1}Y] = 0.$$

It follows from the Euler equation  $\frac{d}{dt}Y = [Y, \text{rot}^{-1}Y]$  that *a field  $Y$  is an equilibrium state of our system (i.e., a stationary motion) iff  $Y$  is a critical point of the kinetic energy  $H(X)$  on the orbit  $\mathcal{O}(X_0)$ .*

This fact was obtained in Arnold's work [2], in which he also found the expression for the second differential of the function  $H(X)$  on the orbit  $\mathcal{O}(X_0)$ .

A divergence-free vector field  $X$  on a three-dimensional manifold  $M$  is called a *Beltrami field* if it is an eigenvector of the vorticity operator:  $\text{rot}X = \lambda X$ ,  $\lambda \in \mathbb{R}$ . The Reeb field  $\xi$  on a contact manifold is an example of Beltrami field,  $\text{rot}\xi = \xi$ . Beltrami fields have a number of remarkable properties. In particular, a Beltrami field is the velocity field of the stationary motion of an ideal incompressible fluid; the Beltrami field  $X$  is a critical point of the kinetic energy  $H(X)$  among all the fields obtained from  $X$  by the action of the diffeomorphism group. The planes orthogonal to the Beltrami field defines a contact structure. By a *Beltrami field* one also means a divergence-free field  $X$  parallel to its vorticity:  $\text{rot}X = fX$ , where  $f$  is a certain function on  $M$ . The topology and the hydrodynamics of the Beltrami fields were studied in [8]-[11]. The topology of stationary flows for which the vorticity vector  $\text{rot}V$  is noncollinear to the velocity field  $V$  almost everywhere was studied in [1], [3], [4], [2].

## 4 Curvature of the group $\mathcal{D}_\mu(M^3)$

For the weak bi-invariant pseudo-Riemannian structure (2.2) on the group  $\mathcal{D}_\mu(M^3)$ , the covariant derivative, the curvature tensor, and the sectional curvatures of the bi-invariant metric have the usual form:

$$\nabla_X^0 Y = \frac{1}{2}[X, Y], \quad (4.1)$$

$$R^0(X, Y)Z = -\frac{1}{4}[[X, Y], Z], \quad (4.2)$$

$$K_\sigma^0 = \frac{1}{4} \langle [X, Y], [X, Y] \rangle_e. \quad (4.3)$$

Taking into account the fact that  $[X, Y] = \text{rot}(Y \times X)$ , where  $\times$  is the vector product on a three-dimensional Riemannian manifold, we obtain the following formula for the sectional curvatures of the bi-invariant metric (2.2): if  $\sigma$  is a plane given by an orthonormal (with respect to (2.2)) pair of vectors  $X, Y \in T_e \mathcal{D}_\mu(M)$ , then

$$K_\sigma^0 = \frac{1}{4} \int_M g([X, Y], Y \times X) d\mu(x).$$

To find the sectional curvatures of the group  $\mathcal{D}_\mu(M^3)$  with respect to the right-invariant weak Riemannian structure, we apply the general formula of the previous section. In our case, it can be simplified. On a three-dimensional manifold  $M$ , the following elementary formulas hold for any vector fields  $X$  and  $Y$  on  $M$ :

$$X \times \text{rot}Y + Y \times \text{rot}X = -(\nabla_X Y + \nabla_Y X) + \text{grad}(X, Y),$$

$$\nabla_X X = \text{rot}X \times X + \frac{1}{2} \text{grad}(X, X).$$

Therefore, the projector  $P = \text{rot}^{-1} \circ \text{rot}$  of the space  $\Gamma(TM)$  of vector fields on  $M$  on the space  $T_e \mathcal{D}_\mu(M)$  of exact divergence-free fields on  $M$  acts as follows:

$$P(\nabla_X Y + \nabla_Y X) = \text{rot}^{-1}([X, \text{rot}Y] + [Y, \text{rot}X]),$$

$$P(\nabla_X X) = \text{rot}^{-1}[X, \text{rot}X].$$



**Theorem 4.1** ([17]). *The sectional curvature of the group  $\mathcal{D}_\mu(M)$  with respect to the right-invariant weak Riemannian structure (2.2) in the direction of a plane  $\sigma$  given by an orthonormal pair of vectors  $X, Y \in T_e \mathcal{D}_\mu(M^3)$  is expressed by the formula*

$$\begin{aligned} K_\sigma = & -\frac{1}{2} \int_M g(X, [[X, Y], Y]) d\mu - \frac{1}{2} \int_M g([X, [X, Y]], Y) d\mu - \\ & - \frac{3}{4} \int_M g([X, Y], [X, Y]) d\mu + \int_M g(\text{rot}^{-1}[X, \text{rot}X], Y \times \text{rot}Y) d\mu - \\ & - \frac{1}{4} \int_M g(\text{rot}^{-1}([X, \text{rot}Y] - [\text{rot}X, Y]), X \times \text{rot}Y - \text{rot}X \times Y) d\mu. \end{aligned} \quad (4.4)$$

If the vector fields  $X$  and  $Y$  are eigenvectors of the operator  $\text{rot}$ ,  $\text{rot}X = \lambda X$ ,  $\text{rot}Y = \mu Y$ , then formula (4.4) takes a simpler form:

$$\begin{aligned} K_\sigma = & -\frac{1}{2} \int_M g(X, [[X, Y], Y]) d\mu - \frac{1}{2} \int_M g([X, [X, Y]], Y) d\mu - \\ & - \frac{3}{4} \int_M g([X, Y], [X, Y]) d\mu - \frac{(\lambda - \mu)^2}{4} \int_M g(\text{rot}^{-1}[X, Y], X \times Y) d\mu. \end{aligned} \quad (4.5)$$

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